REVIEW SESSION

Chapter 2 The Real Numbers

$\mathbb{R}$

field [§2.1]
$+, -, \times, \div$
field axioms

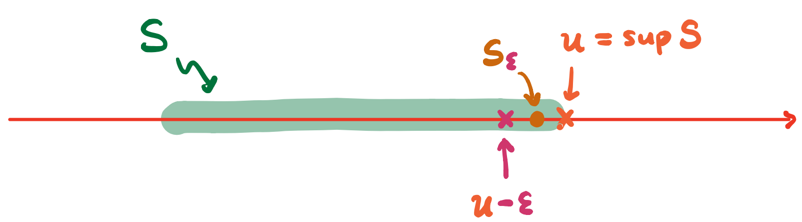
ordered [§2.1–2.2]
"trichotomy"
$a<b,\ a=b,\ a>b$

complete [§2.3–2.4]
existence of
$\sup S,\ \inf S$

Completeness Property [§2.3]: Every $\phi \neq S \subseteq \mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Def$^{\underline{n}}$ [§2.3]: $u = \sup S \iff \begin{cases} u \geq s & \forall s \in S \\ \forall \varepsilon > 0,\ \exists s_\varepsilon \in S & \text{s.t. } u - \varepsilon < s_\varepsilon \end{cases}$

Picture:



! Caution!
$\sup S \in S$
OR $\sup S \notin S$

Useful Inequalities [§2.2]: AM-GM ineq., (reversed) triangle ineq., Bernoulli's ineq.

Useful Facts [§2.4]:
(from completeness)
- $\mathbb{N}$ is NOT bounded above (Archimedean Property)
- Density of $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$
- Existence of $\sqrt{2}$

Intervals [§2.5]:
- characterization of intervals ("Connectedness")
- Nested Interval Property: ("compactness")

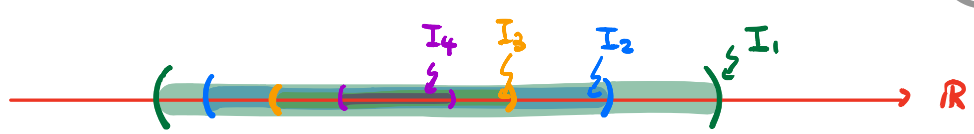
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots \Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$$

closed and bounded intervals

$\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$

$\bigcap_{n=1}^{\infty} [n, \infty) = \phi$

Picture:



Chapter 3 Sequences (and Series)

Other notation: $\{x_n\}_{n=1}^{\infty}$

seq. $(x_n) = (x_1, x_2, x_3, x_4, \ldots\ldots) : \mathbb{N} \to \mathbb{R}$

Def$^{\underline{n}}$: $\boxed{\S 3.1}$

$\lim(x_n) = L \iff \forall\, \varepsilon > 0,\ \exists\, K \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon \quad \forall\, n \geq K$

depends on ε

Limit Thm A: $\boxed{\S 3.2}$ If $\lim(x_n)$ and $\lim(y_n)$ exist, then

- $\lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n)$
- $\lim(x_n y_n) = \lim(x_n)\lim(y_n)$
- $\lim\left(\dfrac{x_n}{y_n}\right) = \dfrac{\lim(x_n)}{\lim(y_n)}$ $\leftarrow$ Provided: $y_n \neq 0$, $\lim(y_n) \neq 0$

$\left(\frac{1}{n}\right) \to 0$

Limit Thm B: $\boxed{\S 3.2}$ If $\lim(x_n)$ and $\lim(y_n)$ exist, then

" $x_n \leq y_n \quad \forall\, n \in \mathbb{N} \implies \lim(x_n) \leq \lim(y_n)$ "

[! Caution ! Only get "$\leq$" even if $x_n < y_n \quad \forall\, n \in \mathbb{N}$. E.g. $0 < \frac{1}{n}$]

FACT: (x_n) convergent $\xRightarrow{\ \S 3.2\ }$ (x_n) bounded

$\Longleftarrow$ + monotone

Monotone Convergence Thm $\boxed{\S 3.3}$

$(x_n) = ((-1)^n)$

$(x_n) = \left(\frac{1}{n}\right)$

$(x_n) = \left(\frac{(-1)^n}{n}\right)$

To show (x_n) divergent

(I) (x_n) unbounded $\boxed{\S 3.2}$

(II) $\exists$ two subseq of (x_n) $\boxed{\S 3.4}$

$$(x_{n_k}) \to L$$
$$\neq$$
$$(x_{m_k}) \to L'$$

To show (x_n) convergent

(I) ε-K definition $\boxed{\S 3.1}$

(II) Limit thms $\boxed{\S 3.2}$

(III) Squeeze thm $\boxed{\S 3.2}$

*(IV) Monotone Convergence Thm $\boxed{\S 3.3}$

*(V) Cauchy criteria $\boxed{\S 3.5}$

do NOT need to know the limit

Def$^{\underline{n}}$: $\boxed{\S 3.5}$

(x_n) is Cauchy $\iff$ $\forall \varepsilon > 0,\ \exists H \in \mathbb{N}$ s.t. $|x_n - x_m| < \varepsilon \quad \forall n, m \geq H$

depends on ε

no relation between them

Cauchy Criteria: $\boxed{\S 3.5}$ (x_n) convergent $\iff$ (x_n) Cauchy

"iff"

Bolzano-Weierstrass Thm: $\boxed{\S 3.4}$ Any bounded seq has a convergent subseq.

[! Caution ! May have different subseq's converging to different limits.
E.g. $((-1)^n)$ $\rightsquigarrow$ limsup & liminf]

Chapter 4 Limits (of functions)

Setup: $f : A \longrightarrow \mathbb{R}$, $c \in \mathbb{R}$ is a cluster pt of A

[! Caution ! Either $c \in A$ or $c \notin A$ is possible E.g.) $A = [0,1)$]

Def$^{\underline{n}}$: $\boxed{\S 4.1}$

$\lim\limits_{x \to c} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon \quad \forall x \in A,\ \underset{x \neq c}{0 < |x - c|} < \delta$

depends on ε

Sequential Criteria: $\boxed{\S 4.1}$

$\lim\limits_{x \to c} f(x) = L \iff \lim(f(x_n)) = L$ $\forall$ seq. (x_n) in $\underset{x_n \neq c}{A \setminus \{c\}}$ s.t. $\lim(x_n) = c$

limit of function

limit of seq.

[FACT: Useful to show $\lim\limits_{x \to c} f(x)$ does NOT exist. E.g.) $f(x) = \sin \frac{1}{x}$]

- Limit Thm A and B carries over from seq. to functions $\boxed{\S 4.2}$

Chapter 5 Continuous Functions

§5.1

Def$^{\underline{n}}$: $f : A \to \mathbb{R}$ is continuous at $c \in A$ $\iff$ $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ $\forall\, x \in A$, $|x-c| < \delta$

depends on ε (and c)

no $0<$

[! Caution! Unlike $\lim_{x \to c} f(x)$, we NEED $c \in A$ here.]

Sequential Criteria: §5.1

$f : A \to \mathbb{R}$ is cts at a cluster pt. $c \in A$ $\iff$ $\lim(f(x_n)) = f(c)$ $\forall$ seq. (x_n) in A s.t. $\lim(x_n) = c$

[FACT: Useful to show Discontinuity at c. E.g.) $f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$]

§5.2

Facts: f, g cts $\Rightarrow$ $f \pm g$, fg, f/g, $f \circ g$

composition

new!

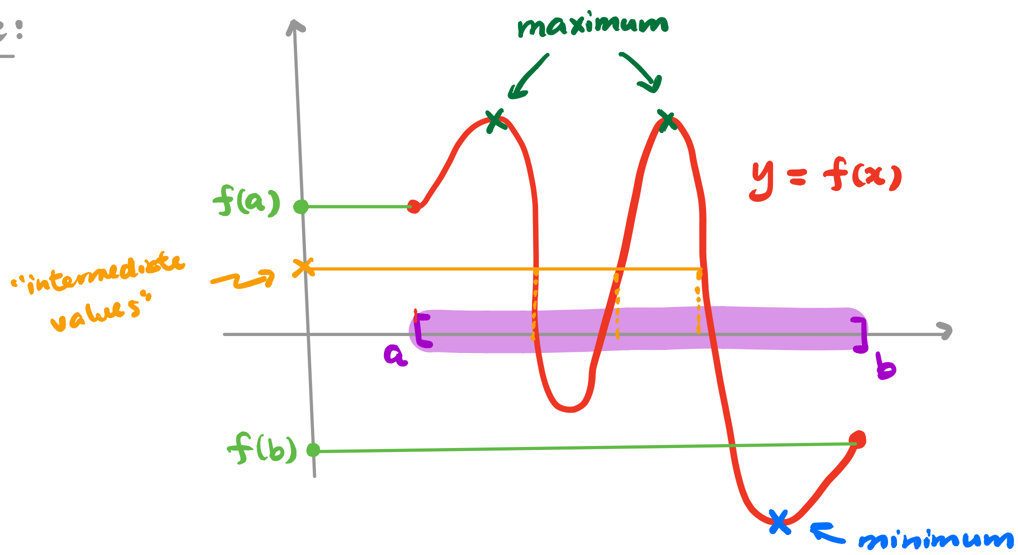
Two Theorems for cts $f : [a,b] \longrightarrow \mathbb{R}$ §5.3

closed + bdd

Extreme Value Thm: f achieves its absolute maximum and minimum.

Intermediate Value Thm: f achieves ALL intermediate values between $f(a)$ and $f(b)$.

Picture:



Def^n: $\boxed{\S 5.4}$

$f: A \to \mathbb{R}$ is uniformly cts (on A) $\iff$ $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

depends ONLY on ε, but NOT u, v

$|f(u) - f(v)| < \varepsilon \quad \forall u, v \in A, \ |u - v| < \delta$

FACT: f unif. cts on A $\Longrightarrow$ f cts on A (i.e. at ALL $c \in A$)

$\not\Longleftarrow$ $\because$ δ may depend on $c \in A$

e.g.) $f(x) = x$

e.g. $f(x) = \frac{1}{x}$

Two Important Thm about uniform continuity $\boxed{\S 5.4}$

Uniform Continuity Thm:

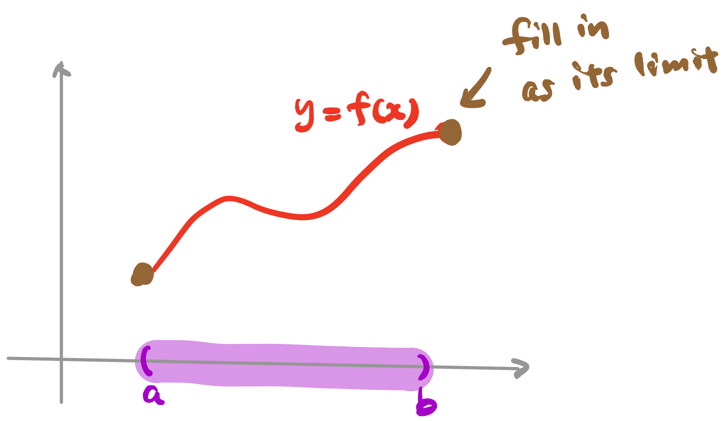
Any cts $f : [a, b] \to \mathbb{R}$ is uniformly cts.

closed + bdd

Continuous Extension Thm:

Any uniformly cts $f : (a, b) \to \mathbb{R}$ can be continuously extended to $[a, b]$.

Picture:



~ END OF REVIEW SESSION ~

Good Luck!